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Partition complexity in a network of chaotic elements

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Abstract. A network of chaotic elements is investigated with the use of globally coupled maps. Elements can split into clusters with synchronized oscillation. In a partially ordered phase, the clustering has an inhomogeneous tree structure like replica symmetry breaking in spin glass. The clustering has large variety by attractors and strongly depends on initial conditions. Variety of clusterings is characterized by the distribution of partitions originated in spin glass theory. Qualitative similarity and quantitative disagreement of our attractors with spin glass are clarified.

1. Introduction

Studies of networks of chaotic elements [1-4] have been growing as a novel paradigm for complex dynamical systems covering neurodynamics [5-8], fluid dynamics, condensed matter such as the Josephson junction array and charge density wave [3, 4], optics [9, 10], evolution dynamics [11] and economics. Among these studies, the 'globally coupled map' (GCM) [1, 2] plays a similar important role as the Sherrington-Kirkpatrick (SK) model played for the spin glass [12].

Here we study the simplest model for network of chaotic elements. It is given by

$$x_{n+1}(i) = (1-\varepsilon) f(x_n(i)) + \frac{\varepsilon}{N} \sum_{j=1}^{N} f(x_n(j)) \tag{1}$$

where n is a discrete time step and i is the index for the elements ($i = 1, 2, \ldots, N =$ system size). We choose here the logistic map $f(x) = 1 - ax^2$, as the simplest model for globally coupled chaotic systems.

The model corresponds to a mean-field version of coupled map lattices (CML), originally proposed as a prototype model for spatio-temporal chaos [13-16]. Our dynamics (1) consists of a parallel nonlinear transformation and a feedback from the 'mean field'.

An important notion in globally coupled dynamical systems is *clustering*. After our system falls on an attractor, elements $x_n(i)$ split into clusters. Here, elements belonging to the same cluster L take exactly an identical value $X_n(L)$. This variable $X_n(L)$ can change in time, but the clustering condition itself is invariant after our system falls on an attractor. Clustering is characterized by the number of clusters k and the number of elements in each cluster, given by $[N_1, N_2, \ldots N_k](\sum_{j=1}^{k} N_j = N)$.

As has been studied in previous papers [1, 2], there are four phases for our model: (i) coherent phase, where a single synchronized attractor ($k = 1$) occupies all basin volumes; (ii) ordered phase, where attractors with a small number of clusters

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($k=o(N)$) occupy basin volumes; (iii) partially ordered (PO) phase, or intermittency phase†, where a variety of attractors with different clusterings coexist—some have a large number of clusters, while others have few; (iv) turbulent phase, where all elements are completely desynchronized ($k=N$).

At the PO phase, our system has many attractors with different clusterings. There are a variety of attractors with different ways of partition of N elements. This partition complexity reminds us of the studies in spin glass (SK model) [17, 18]. Derrida has found universality in this partition complexity, with the use of a distribution function of partitions. We study this distribution in our system.

The paper is organized as follows. In section 2, we introduce some quantifiers to characterize the partition complexity, borrowing from the spin glass problem. The quantifiers include the distribution of cluster numbers and the probability that two elements fall on the same cluster.

In section 3, numerical results of these quantifiers are presented with the emphasis on the enhancement of fluctuation of partition complexity at the PO phase. The distribution of partition complexity has qualitative similarity with the spin glass, although the strength of fluctuation quantitatively differs from the universal form for spin glass. Section 4 is devoted to discussions and summary.

2. Distribution of partitions: definition

In the partially ordered phase, the partition by $[N_1, \ldots, N_k]$ is strongly non-uniform, as has also been seen in the spin glass [12], random energy model, random maps, and fracture [18]. In this PO phase, there are many attractors with different ways of partition to clusters ($k, [N_1, \ldots, N_k]$). To see the variety of the clusterings, we measure a cluster distribution sampled from many initial conditions. Since the clustering characteristics ($k, [N_1, \ldots, N_k]$) has too much information, we study only the distribution of following reduced quantities, derived from the above characteristics.

(1) Cluster number k, or scaled cluster number $c=k/N: Q(c)$, the distribution of c, is calculated from sampling over many initial conditions. From $Q(c)$, average cluster number ratio $\langle c\rangle$ and its variance $\langle(c-\langle c\rangle)^2\rangle$ are derived, which will be studied in detail.

(2) Partition variety Y. To study the non-uniformity of partitions we use the quantity $Y=\Sigma_{j=1}^{k}(N_j/N)^2$, first introduced in the spin glass [12, 17, 18]. Y gives the probability that two arbitrarily chosen elements fall on the same cluster, for a given attractor.

The probability distribution $\pi(Y)$ is calculated from an ensemble of randomly chosen initial conditions. This probability $\pi(Y)$ gives complexity of partition into clusters. If attractors with nearly equal partition to M clusters occupy a large basin volume, $\pi(Y)$ has a large peak slightly above $Y=1/M$. Again, we will study the average $\langle Y\rangle$ and the variance $\langle(\delta Y)^2\rangle \equiv \langle(Y-\langle Y\rangle)^2\rangle$ in detail.

Before showing numerical results, let us briefly look back at the spin glass. In the spin glass, Y is defined as the probability that two arbitrarily chosen initial conditions fall on an identical metastable state. The initial conditions are chosen from all possible 2^N bit configurations. The distribution $\pi(Y)$ is introduced as the distribution of Y

† In our model, the PO and intermittency phases are identical. When we emphasize a dynamical nature, it is called intermittency as in the previous paper [1]. Otherwise we call it the PO phase here. In the previous study the PO phase exists also as a 'glassy' state, at a region between the coherent and ordered phases. Recent numerical studies, however, suggest that this glassy behaviour may be a very long transient [19], which will not be discussed here.

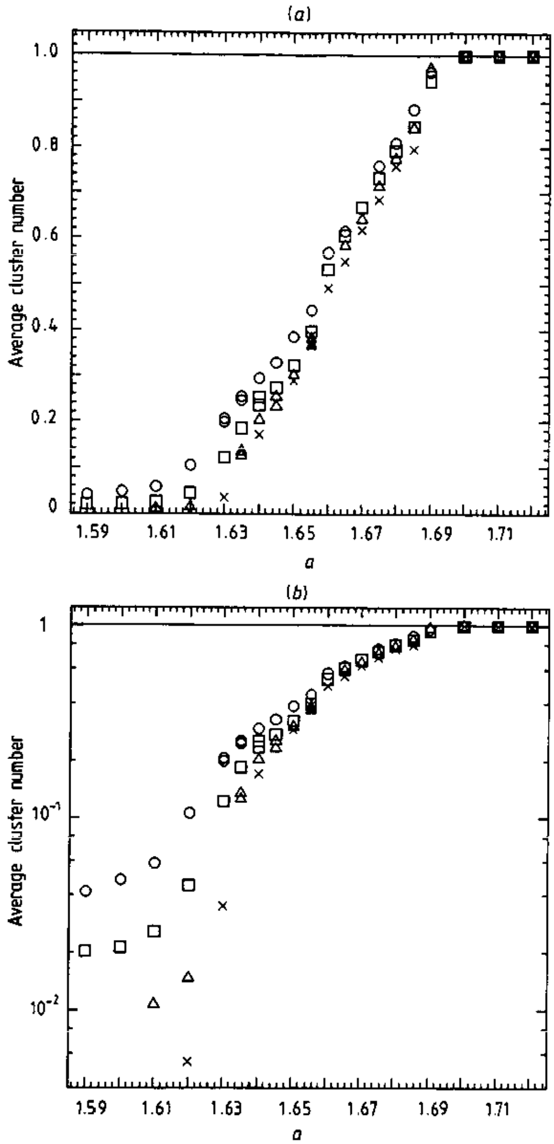

Figure 1. Scaled average cluster number $\langle c\rangle$ as a function of a, for $\varepsilon=0.1$. $N=100$ (○), $N=200$ (□), $N=400$ (△), and $N=800$ (×). All numerical results throughout the present paper (figures 1-10) are obtained from 5000 randomly chosen initial configurations, and after discarding 40 000 steps as transients. To check the accuracy we have sometimes carried out a few runs for different sets of initial conditions, where more than one mark is overlaid. (*a*) Normal plot; (*b*) semi-log plot.

over different samples, with different sets of random couplings. It is shown that $\langle(\delta Y)^2\rangle$ remains finite even in the limit $N\to\infty$, (not 'self-averaging'). Furthermore, there is a universal relationship $\langle(\delta Y)^2\rangle = g(\langle Y\rangle) \equiv \frac{1}{3}\langle Y\rangle(1-\langle Y\rangle)$ [17]. Some other examples like random maps, random energy models, and so on also give the same or close relationship between $\langle(\delta Y)^2\rangle$ and $\langle Y\rangle$ [18, 20].

Our partition variety Y and $\pi(Y)$ are extended from the above definitions for spin glass. Instead of 'two initial conditions' in spin glass we take two elements $x(i)$. In

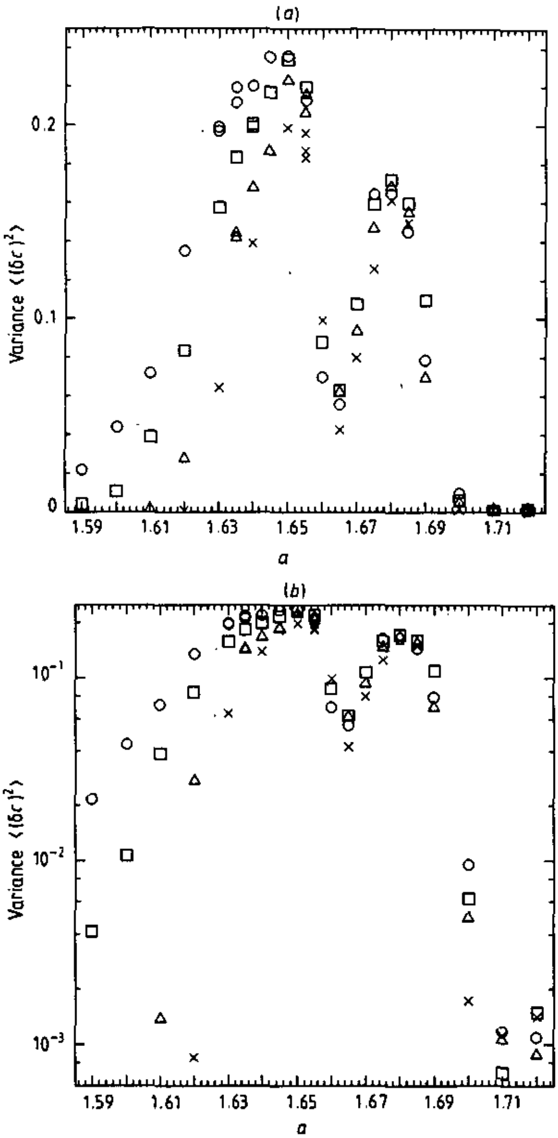

Figure 2. Variance of scaled cluster number $\langle(\delta c)^2\rangle$ as a function of a, for $\varepsilon = 0.1$. $N = 100$ (○), $N = 200$ (□), $N = 400$ (△), and $N = 800$ (×). (*a*) Normal plot; (*b*) semi-log plot.

our case, the sampling is taken over all initial conditions rather than a different choice of couplings. These revisions of definitions are rather natural, since our model does not include extrinsic randomness, but creates randomness through chaos, which can differ by initial conditions.

3. Numerical results for the distribution of partitions

To study the partition complexity in PO phase, we have performed two sets of numerical simulations, changing the nonlinearity a. For the first set, the couplings is fixed at $\varepsilon = 0.1$, while for the latter, it is fixed at $\varepsilon = 0.2$.

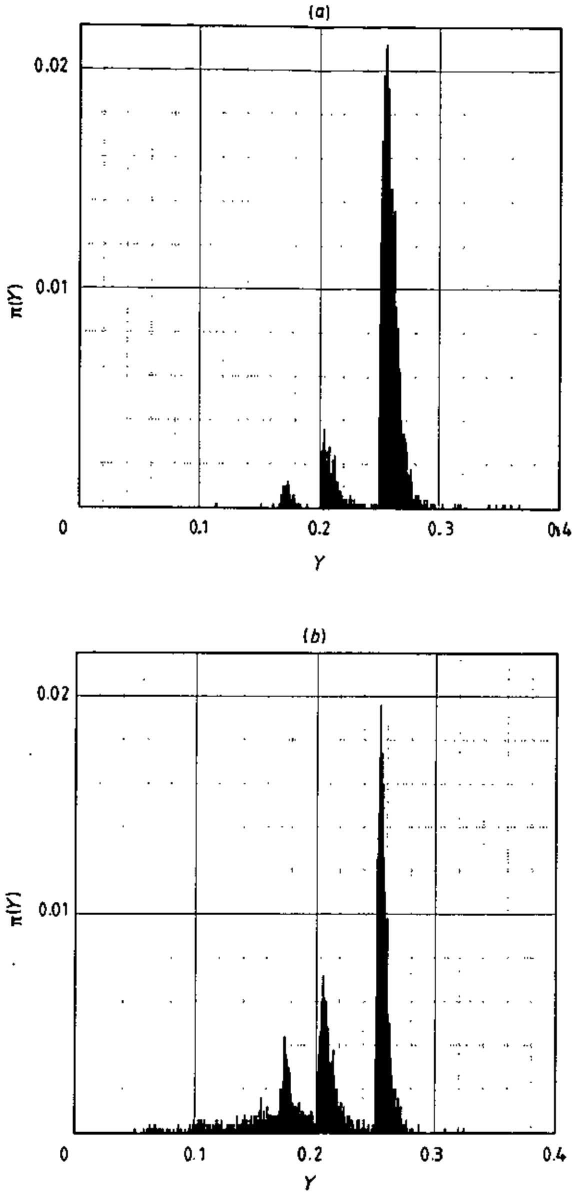

Figure 3. Probability distribution of partitions $\pi(Y)$, calculated from 5000 randomly chosen initial conditions. $\varepsilon = 0.1$, and $N = 200$. (a) $a = 1.6$; (b) $a = 1.62$; (c) $a = 1.64$; (d) $a = 1.66$; (e) $a = 1.68$.

For $\varepsilon = 0.1$, successive transitions occur at $a = a_c(\varepsilon) \approx 1.655$ (ordered → PO), and at $a \approx 1.70$ (PO → turbulent) [1]. From the distribution $Q(c)$, we measure $\langle c \rangle$ and $\langle (\delta c)^2 \rangle$, as are plotted in figures 1 and 2. In the ordered phase, both $\langle c \rangle$ and $\langle (\delta c)^2 \rangle$ decrease towards zero in proportion to $1/N$.

On the other hand, both $\langle c \rangle$ and $\langle (\delta c)^2 \rangle$ seem to approach N-independent values for large N, in the PO phase. The enhancement of fluctuation is clearly seen at the

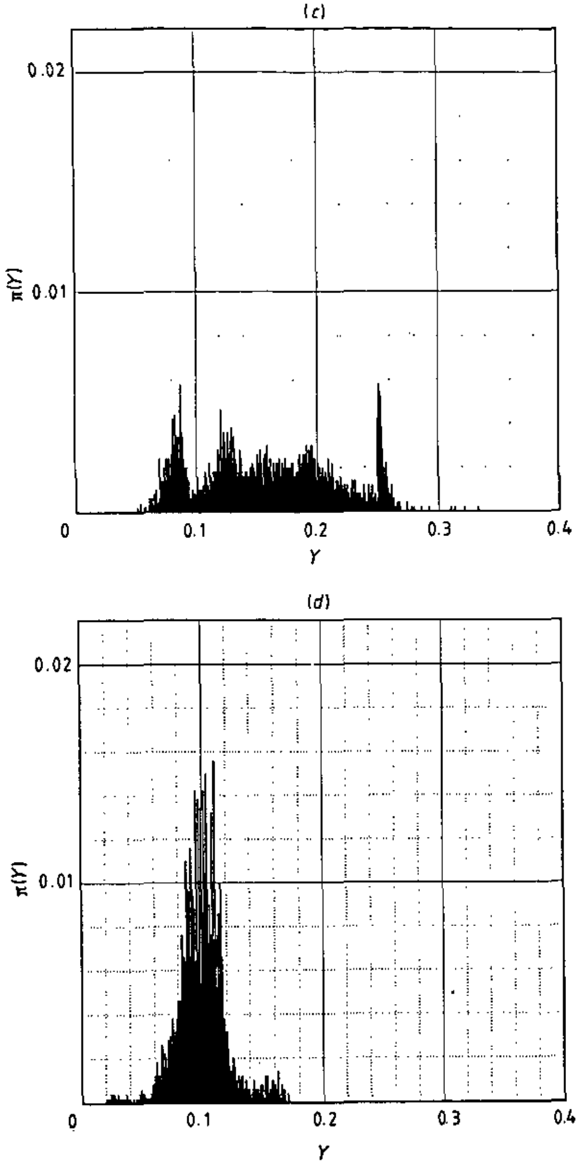

Figure 3. (continued)

transition from the ordered to PO phase (see figure 2(b)). In the turbulent phase, $\langle c \rangle = 1$, and $\langle (\delta c)^2 \rangle = 0$, since all elements take different values, leading to $Q(c) = \delta(c-1)$.

The distribution of partition variety $\pi(Y)$ is plotted in figure 3 for $a = 1.6$, 1.62, 1.64, 1.66, 1.68, and $\varepsilon = 0.1$. As a is increased, peaks located slightly above $1/M$ appear successively with increasing $M(=4, 5, 6, \ldots)$. These peaks have left endpoints at $1/M$ and correspond to M-cluster attractors. The shapes of $\pi(Y)$ at $Y = 1.66$ and 1.68 have similarity with those for SK models and random maps [18], since both have many peaks with left endpoint at $1/M$, and also a broadband spectrum down to $Y = 0$. As a is increased further, the peak at $Y = 1/N \approx 0$ grows (see figure 3(e)), until $\pi(Y)$ approaches $\delta(Y - 1/N) \to \delta(Y)$, at the turbulent phase.

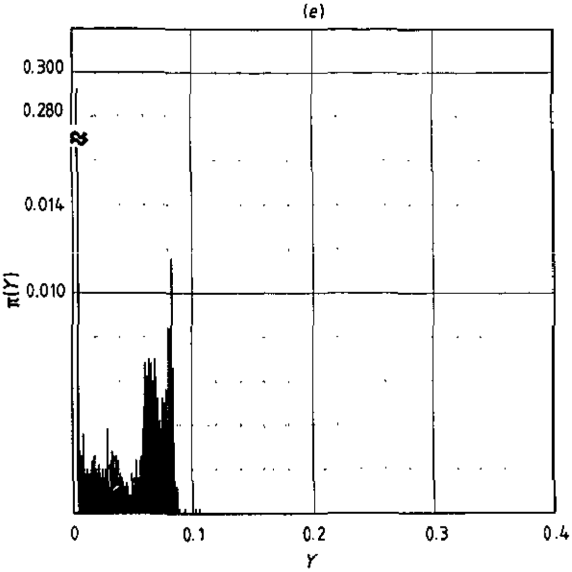

Figure 3. (continued)

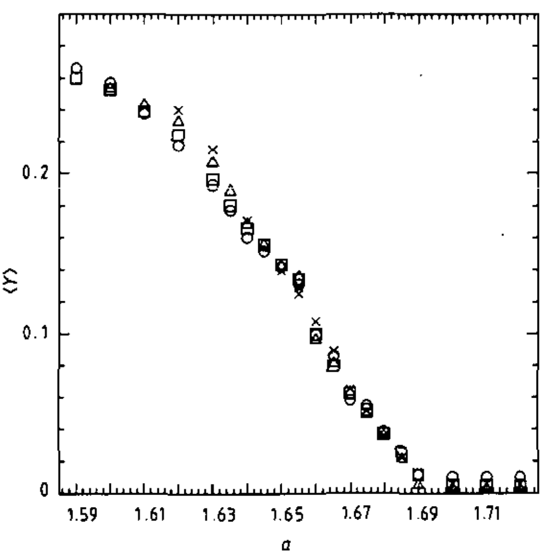

Figure 4. $\langle Y\rangle$ as a function of a, for $\varepsilon=0.1$. $N=100$ (○), $N=200$ (□), $N=400$ (△), and $N=800$ (×).

The average $\langle Y\rangle$ and variance $\langle(\delta Y)^2\rangle=\langle Y^2\rangle-\langle Y\rangle^2$ are plotted in figures 4 and 5. In the ordered phase, $\langle Y\rangle$ is independent of N, while $\langle(\delta Y)^2\rangle$ seems to decrease towards zero with the increasing of N. In the PO phase, both seem to approach N-independent values. The enhancement of fluctuation is again clear. In the turbulent phase, both quantifiers approach zero.

These numerical results are summarized as follows.

(1) The fluctuation $\langle(\delta Y)^2\rangle$ increases as the parameter approaches a_c, the transition point from the ordered to the PO phase. Thus initial condition dependence of partitions

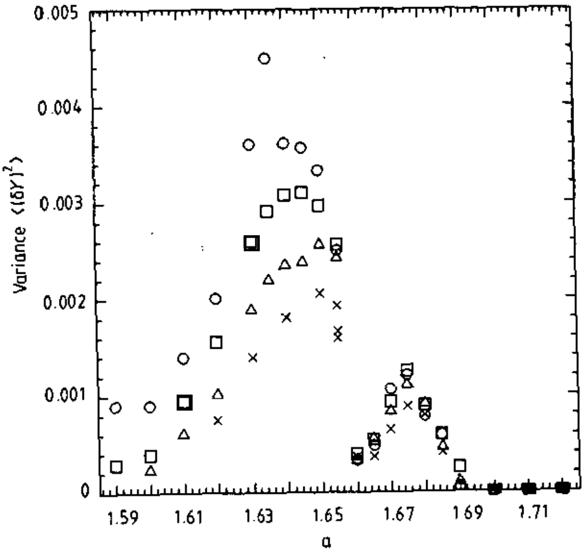

Figure 5. $\langle(\delta Y)^2\rangle$ as a function of a, for $\varepsilon = 0.1$. $N = 100$ (○), $N = 200$ (□), $N = 400$ (△), and $N = 800$ (×).

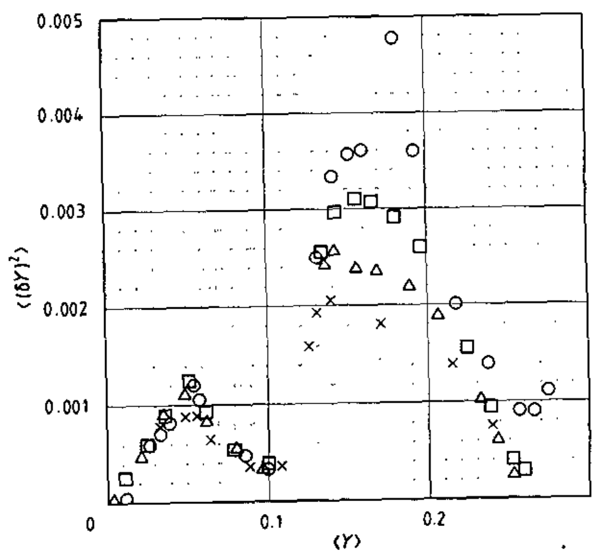

Figure 6. $\langle(\delta Y)^2\rangle$ as a function of $\langle Y\rangle$, for $\varepsilon = 0.1$. $N = 100$ (○), $N = 200$ (□), $N = 400$ (△), and $N = 800$ (×).

is enhanced near $a \approx a_c$. Some initial conditions lead to attractors with many small clusters, while some others lead to those with few, large clusters. The enhancement of fluctuation near $a \approx a_c$ reflects on this increase of variety of partitions.

(2) In the ordered phase, the fluctuation decreases slowly with the size N. This decreasing is roughly fitted by $N^{-\beta}$ with a power $\beta (0 < \beta < 1)$, which depends on a.

(3) In the PO phase, it seems that the fluctuation does not decrease with the system size. In other words, the fluctuation is not self-averaging in the PO phase, as is also found in spin-glass problems.

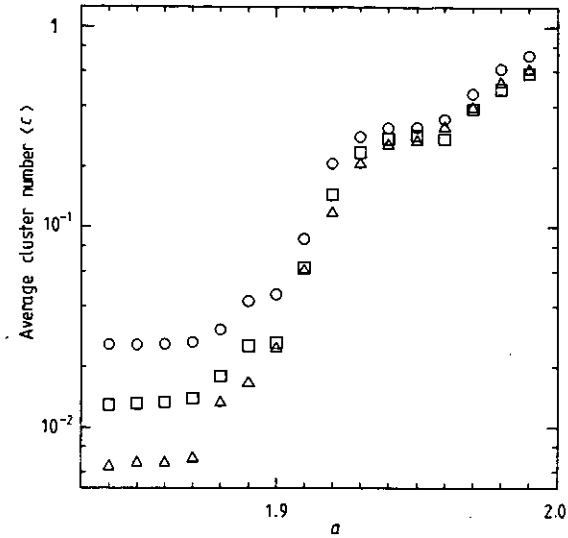

Figure 7. Scaled average cluster number $\langle c \rangle$ as a function of a, for $\varepsilon = 0.2$. $N = 100$ (○), $N = 200$ (□), and $N = 400$ (△).

(4) On figure 6, we have plotted $\langle (\delta Y)^2 \rangle$ as a function of $\langle Y \rangle$. In the PO phase ($0 \leq \langle Y \rangle \leq Y_c \approx 0.1$), it seems that our data for different sizes hit on a single curve. Since our PO phase exists up to $\langle Y \rangle = Y_c$ where $\langle (\delta Y)^2 \rangle$ dies out, it is impossible to fit our curve by the universal functional form $\frac{1}{3}\langle Y \rangle (1 - \langle Y \rangle)$ for the SK model. However, our curve is not too far from the form $\frac{1}{3}\langle Y \rangle (1 - \langle Y \rangle / Y_c)$, which is obtained by replacing the original term $(1 - \langle Y \rangle)$ by $(1 - \langle Y \rangle / Y_c)$, so that the fluctuation dies out at the transition to the ordered phase ($\langle Y \rangle = Y_c$). The form of our curve suggests a possible similarity to our PO phase with the spin glass, even though it does not quantitatively belong to the same universality as the spin glass.

For $\varepsilon = 0.2$, the transition from the ordered to PO phase occurs at $a = a_c \approx 1.9$. Numerical results for the partition complexity are shown in figures 7-10. Most of our conclusions induced from the case with $\varepsilon = 0.1$ are again confirmed, with the following exceptions.

(1) It seems that $\langle (\delta Y)^2 \rangle$ does not decrease with N even in the ordered phase. This remnant fluctuation comes from a nature of the ordered phase here. For $\varepsilon = 0.2$, the basin splits into attractors with the cluster numbers 2 and 3 ($Q(c)$ has peaks at $2/N$ and $3/N$). Both types of attractors have large basin volumes even in the limit of large N, while the cluster number distribution concentrates on $k = 5$ at the ordered phase for $\varepsilon = 0.1$ near $a \approx a_c$.

(2) The fluctuation $\langle (\delta Y)^2 \rangle$ is much smaller than that for $\varepsilon = 0.1$. Numerically it is not easy to get a clear curve for $\langle (\delta Y)^2 \rangle$ as a function of $\langle Y \rangle$. The origin of this reduction of fluctuation is not as yet clear, although its enhancement near $a \approx a_c$ is again clearly seen.

4. Summary and discussion

It is found that there remains a finite fluctuation of partition into clusters, in the partially ordered (or intermittency) phase in globally coupled chaos. In table 1,

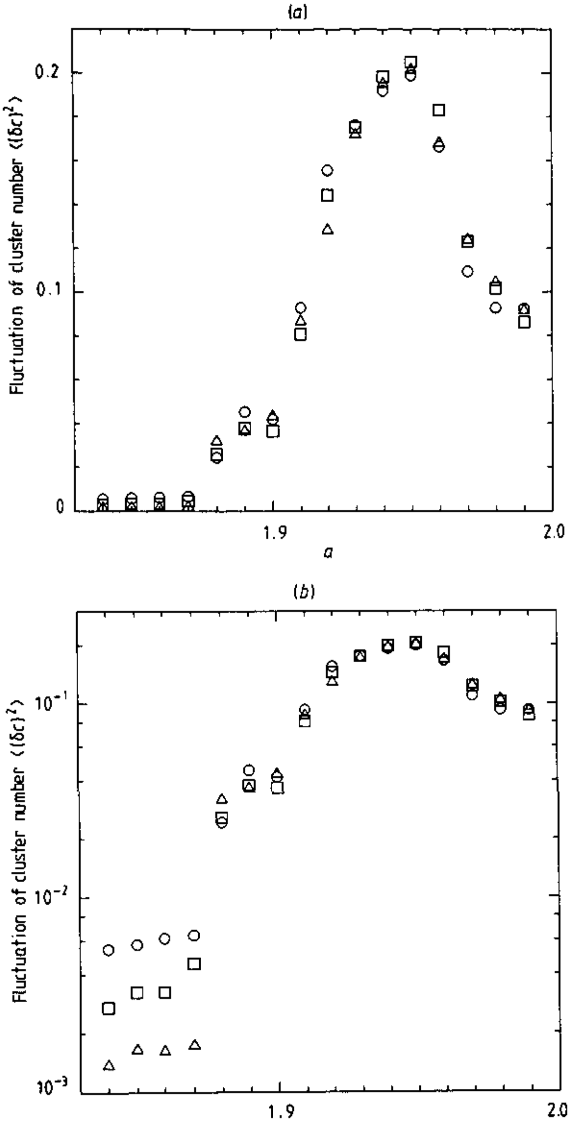

Figure 8. Variance of scaled cluster number $\langle(\delta c)^2\rangle$ as a function of a, for $\varepsilon = 0.2$. $N = 100$ (○), $N = 200$ (□), and $N = 400$ (△). (*a*) Normal plot; (*b*) semi-log plot.

summarized, is the change of the distribution of partitions with the three phases of the globally coupled chaos.

The similarity of our globally coupled map with the spin glass is clarified. It was first noticed in an inhomogeneous tree structure in the clustering [1]. Both in the spin glass and in our system, variety of partition is found for a metastable state or for an attractor respectively, as is quantified by the distribution of cluster number and of the partition variety Y. In our partially ordered phase, the fluctuation is not self-averaging; it does not decay with N, as is also known in the spin glass.

Quantitative deviation from spin glass universality is noted. In our system, randomness is created only through our chaotic dynamics. Qualitatively, this randomness leads

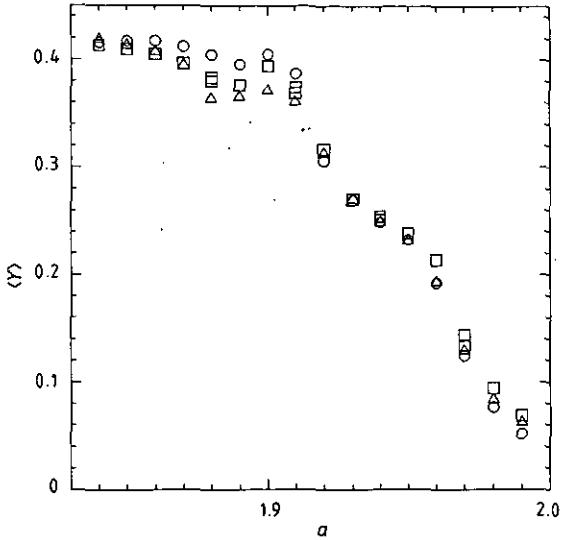

Figure 9. $\langle Y\rangle$ as a function of a, for $\varepsilon=0.2$. $N=100$ (○), $N=200$ (□), and $N=400$ (△).

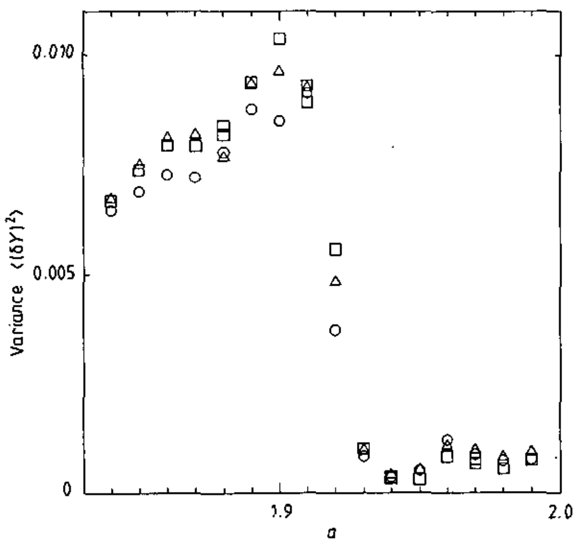

Figure 10. $\langle(\delta Y)^2\rangle$ as a function of a, for $\varepsilon=0.2$. $N=100$ (○), $N=200$ (□), and $N=400$ (△).

to a similar partition complexity as extrinsic randomness, but the quantitative universality no longer holds†.

Inhomogeneous partition leads to a step-like structure of Lyapunov spectra. Lyapunov spectra give how a small disturbance is amplified or reduced. They are averaged logarithms of the eigenvalues of a long-time product of Jacobi matrices $J_n(i,j)=(1-\varepsilon)f'(x_n(i))\delta_{i,j}+(\varepsilon/N)f'(x_n(j))$. If our system falls on a k-cluster attractor

† This kind of quantitative non-universality may be common in a coupled chaotic system. See [21] for non-universality in spatio-temporal intermittency.

Table 1. Features of phases characterized by quantifiers.

	Ordered	Partially ordered	Turbulent
$Q(c)$	$\langle c\rangle \propto 1/N \to 0$ $\langle(\delta c)^2\rangle \propto 1/N \to 0$	$\langle c\rangle$ independent of N $\langle(\delta c)^2\rangle$ independent of N	$\langle c\rangle \approx 1$ $\langle(\delta c)^2\rangle \approx 0$
$\pi(Y)$	$\langle Y\rangle$ independent of N $\langle(\delta Y)^2\rangle \to 0$ with N or indepenent of N	$\langle Y\rangle$ independent of N $\langle(\delta Y)^2\rangle$ independent of N no universal relation between $\langle(\delta Y)^2\rangle$ and $\langle Y\rangle$	$\langle Y\rangle \approx 0$ $\langle(\delta Y)^2\rangle \approx 0$

with the partition $[N_1, N_2, \ldots, N_k]$, the above matrix also splits into k clusters with the same partition. In a typical inhomogeneous clustering, the partition consists of a structure with successively smaller N_ks, like the form $N_k = N \times 2^{-k}$. This structure of the matrix reminds us of the replica symmetry breaking matrix by Parisi [12].

To study the dynamical nature of clusterings, precision-dependent clusterings will be useful [1], where the condition for a cluster is not an exact identity but an identity within a given precision. Two elements i and j are said to belong to the same precision-dependent cluster if they are equal within the precision P. The precision-dependent clustering can change in time. With a finite precision P, we measure the clustering and its corresponding Y value. These quantities can change in time and depend on the precision P. By introducing the distribution of Y over a long time, temporal complexity of partition can be measured for a given attractor. This dynamical complexity will be reported elsewhere.

So far, there is no theory for the statistical mechanics of our globally coupled chaotic system, comparable to the spin glass theory. Construction of it is a future important problem, to open a field for dynamical complexity with large degrees of freedom.

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